Symplectix: Effective computation of the bilinearized Legendrian contact homology
Lecture notes

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Abstract

We recall the definition of Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ and the basic tools to study them. We define the contact homology of such Legendrian knots. We explain linerization processes which allow to extract information from the infinite-dimensional contact homology algebra. Augmentations are auxiliary objects needed for linerization. They turn out to be natural and interesting objects by themselves and we discuss a notion of equivalence of augmentations. We then introduce bilinearized Legendrian contact homology (BLCH), a generalisation of Legendrian contact homology introduced by Bourgeois and Chantraine. The first goal of the talk is to introduce combinatorial methods for the effective computation of BLCH which can be implemented informatically. We then discuss theoretical results obtained by these computational means. The second goal of the talk is to explain that BLCH is a complete invariant for the equivalence of augmentations.

1 Introduction and general context

1.1 The standard contact structure on $\mathbb{R}^3$

Below are reproduced some parts of [5].
In this talk, we will use $(\mathbb{R}^3, \xi_{std})$ where

$$\xi_{std} = \operatorname{span}\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}.$$

ξ_{std} is the kernel of the 1–form $\alpha = dz - y dx$. The plane field is shown in Figure 1.
A *Reeb field* is associated to this contact structure, here it is simply given by $\frac{\partial}{\partial z}$.
Being in $\mathbb{R}^3$ with an "explicit" contact structure will allow to develop combinatorial methods, as it will soon be discussed.

1.2 Legendrian knots and their front projection

A *Legendrian knot* K in $(\mathbb{R}^3, \xi_{std})$ is an embedded S^1 that is always tangent to ξ_{std} :

$$T_x L \in \xi_x, \quad x \in L.$$

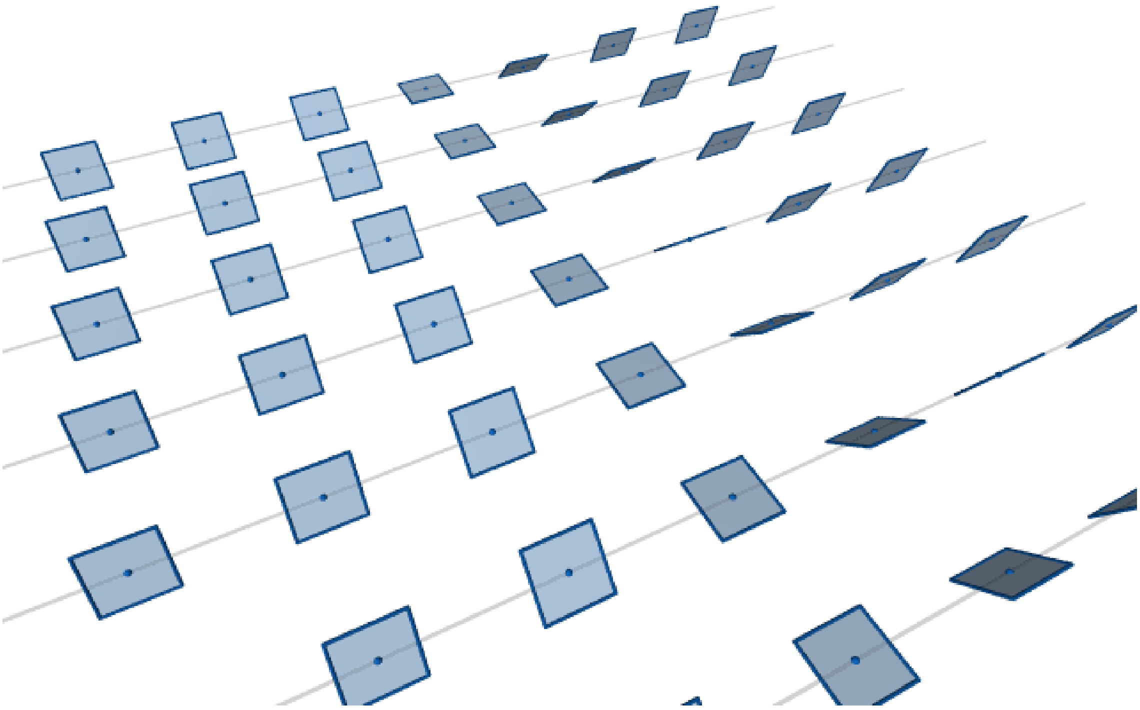

Figure 1: The standard contact structure ξ_{std} (Figure: Patrick Massot[1]).

A classical way to picture Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ is via their *front projection*, defined by the map

$$\Pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, z).$$

The image $\Pi(K)$ of K under Π is called the front projection of K.
A knot diagram represents the front projection of a Legendrian knot iff

1. It has no vertical tangencies;
2. Its only non-smooth points are generalized cusps;
3. At each crossing, the slope of the overcrossing is smaller (that is, more negative) than the undercrossing.

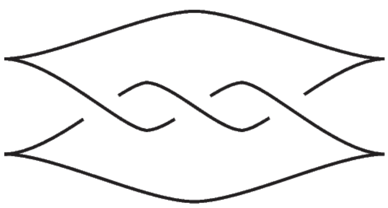

Figure 2: Front projection of the right-handed trefoil (from [5]).

[1] https://www.math.u-psud.fr/~pmassot/exposition/gallerie_contact/

The *Lagrangian projection* is defined by the map

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, y).$$

The image $\pi(K)$ of K under π is called the Lagrangian projection of K.

1.3 Legendrian isotopies

Two Legendrian knots K_0 and K_1 are said to be *Legendrian isotopic* if there is a continuous family $(K_t)_{t\in[0,1]}$ of Legendrian knots starting at K_0 and ending at K_1 (see [5]).

1.4 Ng's resolution procedure

Our goal is to introduce the Legendrian contact homology of a Legendrian knot K.
We will formulate it in terms of the front projection. From a "Legendrian knot theorist" point of view, front projections have two main advantages [2]:

- Existence of Legendrian Reidemeister moves (see [5])

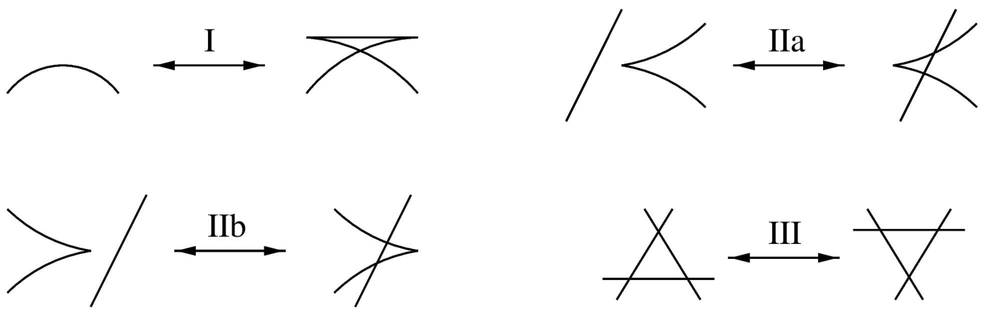

Figure 3: Legendrian Reidemeister moves in front projection (from [13]).

- Appropriate diagrams always correspond to projections of Legendrian knots.

Following Ng [13], we will use the *resolution of a front projection.* In these notes we will not go into details, the interested reader should refer to [13]. Instead, we describe this procedure visually, see figures 4 and 5:
The important property ensured by the resolution procedure is that for all strands i, j,

$$\boxed{z[i] < z[j] \Leftrightarrow \text{slope}[i] < \text{slope}[j]}$$

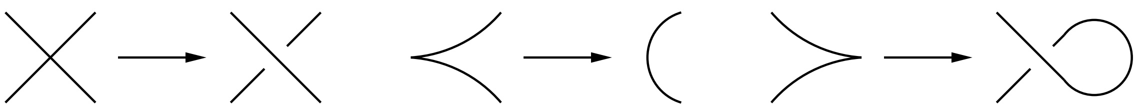

Figure 4: Resolving a front into the Lagrangian projection of a knot (from [13]).

[2]Compared to e.g. Lagrangian projections. Further discussion can be found in Ng's paper [13].

In more geometrical terms, note that

$$\boxed{\begin{aligned} \text{Reeb chords} &\longleftrightarrow \text{crossings in the Lagrangian projection} \\ &\longleftrightarrow \text{crossings } \textit{and right cusps} \text{ in the front projection} \end{aligned}}$$

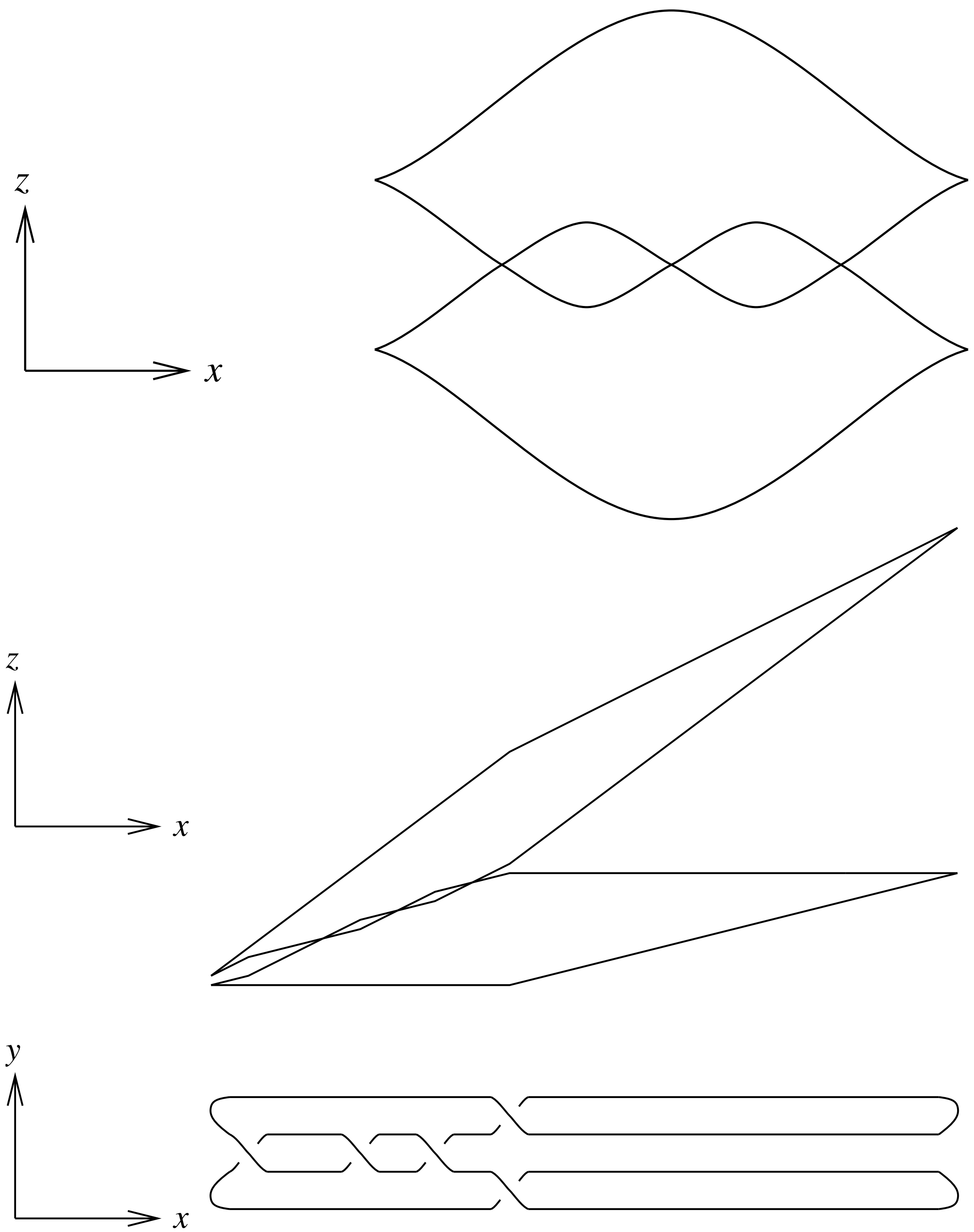

Figure 5: A front projection for the left-handed trefoil (top) is distorted (middle) so that the corresponding Lagrangian projection (bottom), given by $y = dz/dx$, with the same x axis as the middle diagram, is the resolution of the original front (from [13]).

The important property associated to the resolution procedure is the following:

Proposition 1. *The resolution of the front projection of any Legendrian knot K is the Lagrangian projection of a knot Legendrian isotopic to K.*

1.5 Crossings, degrees, Maslov potentials and differentials

This part is heavily based on [11].

- Let K be a Legendrian knot with rotation number zero (see below).

  For computational convenience we assume following Ng [12] that the front of K is *simple*, i.e. that all the right cusps have the same x–coordinate. This can be arranged by a Legendrian isotopy of K.

- We will introduce Chekanov-Eliashberg's *differential graded algebra* (DGA) $(\mathcal{A}, \partial)$ for K. The underlying algebra $\mathcal{A}$ is the free *noncommutative* unital algebra over $\mathbb{Z}_2$ generated by the crossings $c_1, \dots, c_n$ and right cusps $c_{n+1}, \dots, c_{n+r}$ in the front. Thus the elements of $\mathcal{A}$ are finite sums of words in the c_i, where the empty word, denoted by 1, is the identity. The full set $\{c_1, \dots, c_{n+r}\}$ of generators will be denoted by $\mathcal{C}$.

- The grading on $\mathcal{A}$ is defined by assigning an integer degree $|c|$ to each $c \in \mathcal{C}$, and then extending to higher order terms by the rule $|ab| = |a| + |b|$. To define $|c|$, choose a *Maslov potential* μ on the front of K. By definition, μ assigns a real number to each spanning arc in the front in such a way that the upper arc at any cusp is assigned one more than the lower arc; such an assignment can be made consistently if the rotation number of K if 0 ($r(K) = 0$, see [5]). Now set $|c| = 1$ if c is a cusp, and

  $$|c| = \mu(\alpha) - \mu(\beta)$$

  if c is a crossing, where α is the upper arc (the one with a smaller slope) at the crossing and β is the lower arc.

- Finally the *differentials* ∂c for $c \in \mathcal{C}$ are defined by counting suitable immersed disks. The differential is then extended to all of $\mathcal{A}$ by the Leibniz rule and linearity, setting $\partial 1 = 0$. We will not go into the formal definition of ∂c and instead refer the reader to [3] or [5].

- With the suitable definitions, ∂ lowers degree by 1, and $\partial^2 = 0$. This leads to a graded homology $H_*(\mathcal{A}, \partial)$, which coincides with the contact homology of $(\mathbb{R}^3, \xi_{std})$ relative to K (see [6]) and is invariant under Legendrian isotopy of K.

1.6 Augmentations

The algebra $(\mathcal{A}, \partial)$ is infinite dimensional (over $\mathbb{Z}_2$) and not very convenient to use or compute. However, useful information can be extracted from its finite dimensional quotients.

Definition 2. *An augmentation of $(\mathcal{A}, \partial)$ (or "augmentation of K") is an algebra map*

$$\varepsilon : \mathcal{A} \to \mathbb{Z}_2$$

that vanishes on elements of nonzero degree and satisfies $\varepsilon\partial = 0$.

The generators $c \in \mathcal{C}$ with $\varepsilon(c) = 1$, which are all crossings of degree zero, will be called the *augmented crossings* of ε.

1.7 Linearized contact homology

The differential of generators can always be written as

$$\partial c_i = \sum_{l \geq 0} \sum_{1 \leq i_1, \cdots, i_l \leq k} M^i_{i_1 \cdots i_l} c_{i_1} \cdots c_{i_l} \tag{1}$$

for some finite number of coefficients $M^i_{i_1 \cdots i_l} \in \mathbb{Z}_2$.

Definition 3 [Linearization].
*The **linearization** ∂^ε is a differential on the graded $\mathbb{Z}_2$ **vector space** freely generated by $\mathcal{A}$. It is defined by*

$$\partial^\varepsilon c_i = \sum_{l \geq \mathbf{1}} \sum_{1 \leq i_1, \cdots, i_l \leq k} M^i_{i_1 \cdots i_l} \sum_{j=1}^{l} \varepsilon(c_{i_1}) \cdots \varepsilon(c_{i_{j-1}}) c_{i_j} \varepsilon(c_{i_{j+1}}) \cdots \varepsilon(c_{i_l}) \tag{2}$$

i.e. obtained from ∂ by applying the following "linearization procedure"

$$c_{i_1} \cdots c_{i_l} \longmapsto \sum_{j=1}^{l} \varepsilon(c_{i_1}) \cdots \varepsilon(c_{i_{j-1}}) c_{i_j} \varepsilon(c_{i_{j+1}}) \cdots \varepsilon(c_{i_l})$$

Theorem 4 [Chekanov 2002, [3]].
*∂^ε **is a differential** and the set*

$$\left\{ LCH^\varepsilon(K) := H(\mathcal{A}, \partial_\varepsilon) \mid \varepsilon \text{ augmentation of } (\mathcal{A}, \partial) \right\}$$

is invariant under Legendrian isotopy.

The *Chekanov polynomial* $P_{K,\varepsilon}$ is the Poincaré polynomial of this complex,

$$P_{K,\varepsilon}(t) = \sum_{k\in\mathbb{Z}} \dim(H_k)\, t^k$$

where $H_k = \ker \partial_k^\varepsilon / \operatorname{im} \partial_{k+1}^\varepsilon$.
The "big picture" is the following:

$$\boxed{\begin{aligned} \text{Chekanov-Eliashberg DGA} &\xrightarrow{\varepsilon} \text{linearized chain complex with differential } \partial_\varepsilon \\ &\longrightarrow \text{linearized contact homology } LCH^\varepsilon \\ &\longrightarrow \text{Poincaré polynomial } P_{K,\varepsilon}(t) \end{aligned}}$$

1.8 Example of the trefoil

This part was adaptated from [5].
Consider the Legendrian right handed trefoil whose front projection is shown in figure 2.
Let's denote the right cusps by a_1 and a_2 from top to bottom and the crossings by b_1, b_2, b_3 from left to right.
The algebra has five generators a_1, a_2, b_1, b_2, b_3. Their gradings are

$$|a_i| = 1, \quad |b_i| = 0.$$

One easily computes

$$\begin{aligned} \partial a_1 &= 1 + b_1 + b_3 + b_1 b_2 b_3 \\ \partial a_2 &= 1 + b_1 + b_3 + b_3 b_2 b_1 \\ \partial b_i &= 0. \end{aligned}$$

The equation that an augmentation must satisfy is

$$\boxed{0 = 1 + \varepsilon(b_1) + \varepsilon(b_3) + \varepsilon(b_1)\varepsilon(b_2)\varepsilon(b_3)}$$

It has 5 solutions for $\varepsilon(b_1), \varepsilon(b_2), \varepsilon(b_3) \in \{0, 1\}$.

1.9 Equivalence of augmentations

Several notions of equivalence for augmentations of DGAs were introduced in the literature and used in the context of the Chekanov-Eliashberg DGA. It turns out that the equivalence relation among augmentations that controls best the behavior of BLCH (see the next section) is the notion of DGA homotopic augmentations [14, Definition 5.13].

Definition 5 [$(\varepsilon_1, \varepsilon_2)$-DERIVATION].
Let $\varepsilon_1, \varepsilon_2$ be two augmentations of the DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}_2$. A linear map $T : \mathcal{A} \to \mathbb{Z}_2$ is said to be an $(\varepsilon_1, \varepsilon_2)$-derivation if $T(ab) = \varepsilon_1(a)T(b) + T(a)\varepsilon_2(b)$ for any $a, b \in \mathcal{A}$.

Definition 6 [DGA HOMOTOPIES FOR AUGMENTATIONS].
We say that ε_1 is DGA homotopic to ε_2, and we write $\varepsilon_1 \sim \varepsilon_2$, if there exists an $(\varepsilon_1, \varepsilon_2)$-derivation $T : \mathcal{A} \to \mathbb{Z}_2$ of degree $+1$ such that $\varepsilon_1 - \varepsilon_2 = T \circ \partial$.
DGA homotopy is an equivalence relation [7, Lemma 26.3].

It can be shown that

$$\varepsilon_1 \sim \varepsilon_2 \Rightarrow LCH^{\varepsilon_1}(K) = LCH^{\varepsilon_2}(K)$$

so the "real" invariant to consider is rather

$$\left\{ LCH^{[\varepsilon]}(K) \mid \varepsilon \text{ augmentation of } (\mathcal{A}, \partial) \right\}$$

where $[\varepsilon]$ is the equivalence class of ε for DGA homotopy.

Remark 7 : LCH can therefore provide information about equivalence of augmentations, since

$$LCH^{\varepsilon_1}(K) \neq LCH^{\varepsilon_2}(K) \Rightarrow \varepsilon_1 \nsim \varepsilon_2$$

We will see soon that BLCH in in fact the right tool to detect equivalence of augmentations.

2 Bilinearized contact homology

2.1 Motivation

Bilinearized contact homology (BLCH) was introduced by Frédéric Bourgeois and Baptiste Chantraine in [1], following discussions with Petya Pushkar.
We start with a basic observation: the Chekanov-Eliashberg DGA is a *noncommutative* algebra. However, the linearization process loses all this "noncommutative" information since both c_1c_2 and c_2c_1 are linearized as $\varepsilon(c_2)c_1 + \varepsilon(c_1)c_2$.
BLCH sees the non-commutativity of the algebra *by using two augmentations instead of one.*
We will see that LCH is a special case of BLCH by using twice the same augmentation and that **BLCH is a complete invariant for DGA homotopy of augmentations.**

2.2 Definition and first properties

Due to the combinatorial approach taken in the talk, we will introduce BLCH in a rather formal (symbolic!) way. Natural geometrical interpretations do exist, we refer the reader to [1] and to Frédéric Bourgeois himself to hear more about this.

We recall that the differential of generators of the DGA was given by equation (1) and the linearization process by equation (2).
Given *two* augmentations ε_1 and ε_2, we can proceed similarly:

Definition 8 [BILINEARIZATION].
*Given two augmentations ε_1 and ε_2, the **bilinearization** $\partial^{\varepsilon_1,\varepsilon_2}$ is a differential on the graded $\mathbb{Z}_2$ **vector space** freely generated by $\mathcal{A}$. It is defined by*

$$\partial^{\varepsilon_1,\varepsilon_2} c_i = \sum_{l \geq 1} \sum_{1 \leq i_1, \cdots, i_l \leq k} M^i_{i_1 \cdots i_l} \sum_{j=1}^{l} \varepsilon_1(c_{i_1}) \cdots \varepsilon_1(c_{i_{j-1}}) c_{i_j} \varepsilon_2(c_{i_{j+1}}) \cdots \varepsilon_2(c_{i_l}) \tag{3}$$

i.e. obtained from ∂ by applying the following "bilinearization procedure"

$$c_{i_1} \cdots c_{i_l} \longmapsto \sum_{j=1}^{l} \varepsilon_1(c_{i_1}) \cdots \varepsilon_1(c_{i_{j-1}}) c_{i_j} \varepsilon_2(c_{i_{j+1}}) \cdots \varepsilon_2(c_{i_l})$$

This gives rise to an homological notion:

$$BLCH^{\varepsilon_1,\varepsilon_2}(K) := H(\mathcal{A}, \partial^{\varepsilon_1,\varepsilon_2})$$

We have the same kind of results as for LCH (see [1]):

$$\varepsilon_1 \sim \varepsilon_1', \varepsilon_2 \sim \varepsilon_2' \Rightarrow BLCH^{\varepsilon_1,\varepsilon_2}(K) = BLCH^{\varepsilon_1',\varepsilon_2'}(K)$$

Theorem 9 [BOURGEOIS-CHANTRAINE 2012, [1]].
*$\partial^{\varepsilon_1,\varepsilon_2}$ **is a differential** and the set*

$$\left\{ BLCH^{[\varepsilon_1],[\varepsilon_2]}(K) \mid \varepsilon_1, \varepsilon_2 \text{ augmentations of } (\mathcal{A}, \partial) \right\}$$

is invariant under Legendrian isotopy.

Remark 10 : An easy but important remark is to note that

$$LCH^{\varepsilon}(K) = BLCH^{\varepsilon,\varepsilon}(K)$$

so that

$$BLCH^{\varepsilon_1,\varepsilon_2}(K) = BLCH^{\varepsilon_2,\varepsilon_1}(K) = LCH^{\varepsilon_1}(K) = LCH^{\varepsilon_2}(K)$$

whenever $\varepsilon_1 \sim \varepsilon_2$.

Just as for LCH, we define $P_{K,\varepsilon_1,\varepsilon_2}$, the Poincaré polynomial of this complex, as

$$P_{K,\varepsilon_1,\varepsilon_2}(t) = \sum_{k\in\mathbb{Z}} \dim(H_k)\, t^k$$

where $H_k = \ker \partial_k^{\varepsilon_1,\varepsilon_2} / \operatorname{im} \partial_{k+1}^{\varepsilon_1,\varepsilon_2}$.

2.3 Criteria for the equivalence of augmentations

Computer-assisted inspection of several examples led to a striking conjecture about the behaviour of BLCH with respect to equivalence of augmentations. This conjecture was then proved soon after, we may therefore call it a theorem:

Theorem 11 [Bourgeois-G. 2018, [2]]. *Let $\varepsilon_1, \varepsilon_2$ be two augmentations of a Legendrian knot K. Let $P_{K,\varepsilon_1,\varepsilon_2}(t)$ and $P_{K,\varepsilon_2,\varepsilon_1}(t)$ denote the BLCH Poincaré polynomials. Then,*

- *If $\varepsilon_1 \sim \varepsilon_2$, there exist coefficients $(c_i)_{i\in\mathbb{Z}}$ such that*

$$P_{K,\varepsilon_1,\varepsilon_2}(t) = P_{K,\varepsilon_2,\varepsilon_1}(t) = t + \sum_{i\in\mathbb{Z}} c_i t^i$$

  with $c_i = c_{-i}$ for all $i \in \mathbb{Z}$.

- *If $\varepsilon_1 \nsim \varepsilon_2$, there exist coefficients $(c_i)_{i\in\mathbb{Z}}$ such that*

$$P_{K,\varepsilon_1,\varepsilon_2}(t) = 1 + \sum_{i\in\mathbb{Z}} c_i t^i \quad \text{and} \quad P_{K,\varepsilon_2,\varepsilon_1}(t) = 1 + \sum_{i\in\mathbb{Z}} c_i t^{-i}$$

In particular, checking if the t coefficient of $P_{K,\varepsilon_1,\varepsilon_2}(t)$ is equal to the t^{-1} coefficient of $P_{K,\varepsilon_2,\varepsilon_1}(t)$ determines whether $\varepsilon_1 \sim \varepsilon_2$.

Remark 12 : Actually, this theorem can be formulated in much greater generality, and the preceding result makes sense and is proved in higher dimensions as well.

2.4 Geography questions

After proving the criteria for the equivalence of augmentations, the second goal of [2] is to characterize polynomials which can be realized as BLCH polynomials of non DGA homotopic augmenations.

Definition 13 [BLCH-admissible polynomials].
In dimension 3, a Laurent polynomial P with integral coefficients is said to be BLCH-admissible if it can be written as 1 plus a Laurent polynomial with integral coefficients such that its value at -1 is even.

Theorem 14 [Bourgeois-G. 2018, [2]].
Poincaré polynomials realizable as BLCH polynomials of non DGA homotopic augmentations are exactly BLCH-admissible polynomials.

We refer the interested reader to [2] for precise statements and proofs in a more general setting. In particular, the geography is given in all dimensions.

2.5 BLCH matrix

A convenient way to speak of BLCH is via the "BLCH matrix" defined by

$$\Big(P_{K,[\varepsilon_1],[\varepsilon_2]}(t)\Big)_{[\varepsilon_1],[\varepsilon_2]\text{ distinct equivalence classes of augmentations}}$$

To build the BLCH matrix in practice, one proceeds as follows:

1. Compute all polynomials $P_{K,\varepsilon_1,\varepsilon_2}(t)$ for all couples of augmentations $(\varepsilon_1,\varepsilon_2)$, e.g. by putting them in a big matrix $\Big(P_{K,\varepsilon_1,\varepsilon_2}(t)\Big)_{\varepsilon_1,\varepsilon_2}$;
2. Determine the equivalence classes of augmentations;
3. Compress the big matrix by collapsing equivalence classes of augmentations.

Note that such matrices are only defined up to the order of the augmentations (i.e. up to swaps of rows and columns). Having this in mind, the previous results can be restated as follows:

The BLCH matrix is a Legendrian knot invariant.

One of the goals of the following sections is to propose an

Effective algorithm to compute the BLCH matrix.

2.6 2-copies and BLCH

Definition 15 [2-copies of Legendrian knots].
Given a Legendrian knot K in $(\mathbb{R}^3,\xi_{std})$, its 2-copy is the Legendrian link obtained by taking K and a copy of K shifted slightly in the z direction.

We will denote the two components of such a 2-copy by $K1$ and $K2$ respectively, see figure 6. To each crossing on the diagram of a knot correspond 4 crossings on the diagram of its 2-copy, of 4 types: $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$, see figure 7.
Crossings of type $(1,1)$ and $(2,2)$ will be called *pure*, those of type $(1,2)$ and $(2,1)$ will be called *mixed*.
Gradings of crossings of the two components give rise to gradings of pure crossings (or we can directly introduce a Maslov potential on the 2-copy).

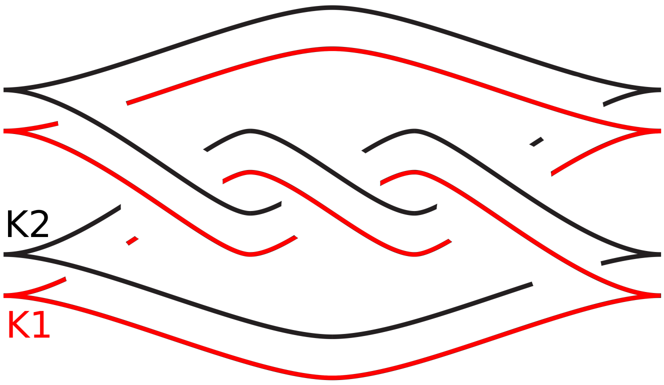

Figure 6: A 2-copy of the right handed trefoil (modified from [5]).

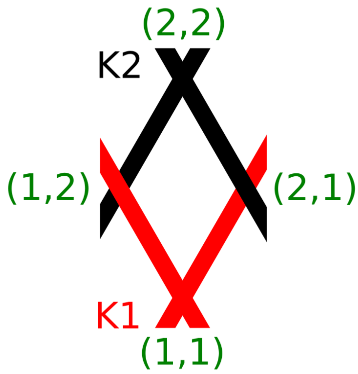

Figure 7: Types of crossings.

Given two augmentations ε_1 and ε_2 of K, we can build an augmentation $\varepsilon_1 \star \varepsilon_2$ on its 2-copy by using augmentation ε_1 for crossings of type $(1,1)$, ε_2 for crossings of type $(2,2)$, and by sending mixed crossings to 0. This is useful due to the following result:

Proposition 16. *Let K be a Legendrian knot and $\varepsilon_1, \varepsilon_2$ be two augmentations of K. Let $\varepsilon_1 \star \varepsilon_2$ be an augmentation defined on its 2-copy as above. Then*

$$LCH^{\varepsilon_1 \star \varepsilon_2}(K1 \cup K2) \cong LCH^{\varepsilon_1}(K) \oplus LCH^{\varepsilon_2}(K) \oplus BLCH^{\varepsilon_1,\varepsilon_2}(K) \oplus \widetilde{BLCH}^{\varepsilon_2,\varepsilon_1}(K)$$

Remark 17 : $\widetilde{BLCH}^{\varepsilon_2,\varepsilon_1}(K)$ is related to $BLCH^{\varepsilon_2,\varepsilon_1}(K)$ but is not indentical to it. What matters for us is the part consisting of $BLCH^{\varepsilon_1,\varepsilon_2}(K)$.

Remark 18 : Morally, "BLCH can be seen as LCH on the 2-copy with suitable augmentations". So BLCH is somehow "both a generalization and a special case of LCH"!

In the next section, we will discuss combinatorial methods which will allow to compute LCH from a given front diagram. This will suffice to compute BLCH since handling 2-copies, types of crossings, building $\varepsilon_1 \star \varepsilon_2$, ... do not present real computational challenges but only some slightly tedious but straightforward manipulations.

3 Combinatorial methods

3.1 Summary and goals

- We now have an interesting homological invariant defined for Legendrian knots that we want to further investigate.
- We have seen that the invariant can be described in terms of the front projection of the knot in a suitable "normal form". With effective computations as a target, we notice that these normal forms can naturally be described as braids, which can themselves be encoded as a vector of integers denoting the crossing of the various braids. This is particularly nice for computational applications:

Front projections in normal forms are naturally encoded as vectors of integers.

  For an example of such an encoding, see figure 10 and the box after it.
- Along the way, we have encountered an interesting class of objects: **augmentations**. These can be naturally seen as subsets of the set of degree 0 generators of the DGA (augmentations are determined by their set of augmented generators).
- We have introduced notions of linearized and bilinearized homologies. We have seen that, up to some combinatorial manipulations, it suffices to compute LCH. This will allow to compute BLCH and then to build the BLCH matrix.

We want to have **effective** methods which can achieve the following:

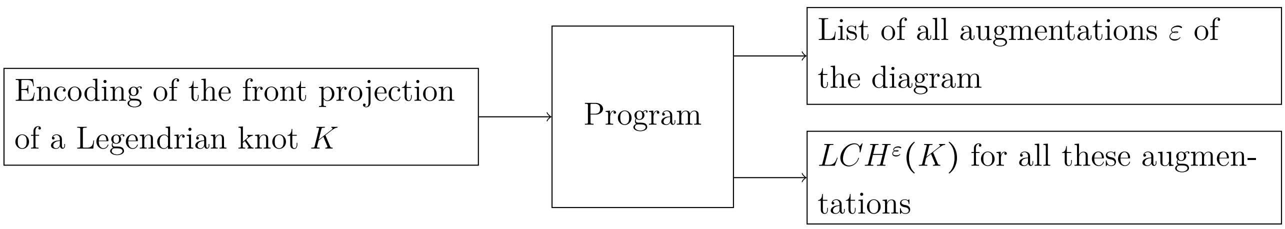

Figure 8: The effectiveness we are looking for.

As we will see, having such methods will suffice (up to some combinatorial work) to answer the problem of equivalence of augmentations and to compute the BLCH matrix.

3.2 A few words about Morse complex sequences and their history

Remark 19 : We will be very sketchy in this section. Instead of trying to give precise definitions of Morse complex sequences (MCS), we will dicuss why and how they were introduced and try to give an idea of what MCS are.

We refer the interested reader to [9] (and also [8], [10]) for precise definitions and proofs. This section is based heavily on [9].

Morse complex sequences were introduced by Petya Pushkar as a combinatorial substitute for generating families (though he did not publish most of his original ideas and is instead credited[3] in [8], which is the first paper to define MCS). They were further developed mostly by Brad Henry and Dan Rutherford ([8], [9], [10]).

The differential topological motivation behind MCS comes from generating families. Technical difficulties arise when trying to rigourously define and use a generating family DGA. MCS use a combinatorial approach instead. This allows to prove theorems by a "hands-on anaysis" (using the terms of [9]), avoiding technical difficulties.

There is a natural equivalence relation on the set of MCSs.

The main result of [10] states, roughly speaking, that

Theorem 20 [Henry-Rutherford 2014, [10]]**.** *For any Legendrian knot K, there is a bijection between equivalence classes of Morse complex sequences and equivalence classes of augmentations (for DGA homotopy).*

3.3 Effectiveness

MCS are built from front projections in standard from of Legendrian knots. The results of Henry and Rutherford give effective bijections from combinatorial objects (MCS in their various "standard forms"), thereby fulfilling the requirements shown in figure 8.

Once the augmentations are found and the differentials are computed, we are done. Indeed, computationally speaking "homology is just linear algebra" which is treated using standard libraries.

4 Effective computation of BLCH for Legendrian knots

4.1 Putting everything together

Morse complex sequences and the associated procedures allow to find augmentations and to compute LCH from the encoding of the front projection (in normal form) of a Legendrian knot. Combining this with the previous sections (mostly the section 2.6), we obtain:

[3]From [8]: "In emails to D. Fuchs in 2001 and 2008, Pushkar outlines his "Spring Morse theory" (. . .) *The ideas behind Morse complex sequences and the equivalence relation we define in this article originate with Petya Pushkar."*

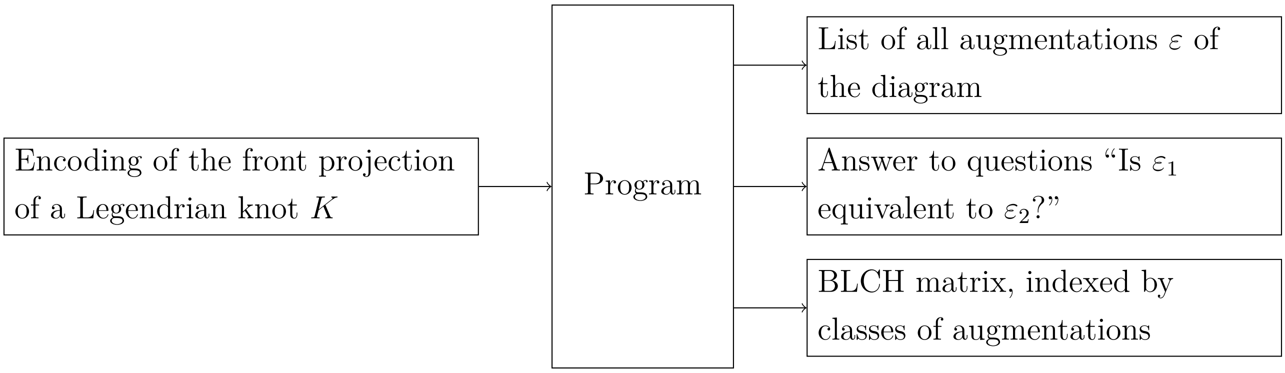

Figure 9: What MCS give us, computationally speaking (after some extra processing and using the “2-copy trick”).

4.2 First results

Now we are able to run the program on all examples from a table of knots.
This provides very useful data to investigate properties of BLCH. For instance, this allowed to find an example of a knot whose BLCH matrix is not symmetric.
Such an example was not known before (see for example what the BLCH matrix of the right-handed trefoil looks like). This example turned out to have a very interesting behaviour with respect to BLCH, and we discuss it in the section 4.3.

Remark 21 : At least [11] and [4] had already done computer-based computations with Legendrian knots. The introduction of [4] provides a striking list of “by-products” of computational methods.

4.3 $m(8_{21})$

In this section (adaptated from [2]), we further investgate the “first interesting example from the list of knots” (according to *several different criteria*).
Let K denote the Legendrian knot (already studied by Melvin and Shrestha[4] in [11, Section 3]), which is topologically the mirror image of the knot 8_{21}, and illustrated in Figure 10.

[4]It is interesting to remark that their 2005 paper is entitled “The nonuniqueness of Chekanov polynomials of Legendrian knots”. Somehow this example is also “the first interesting one for the diagonal of the BLCH matrix”.

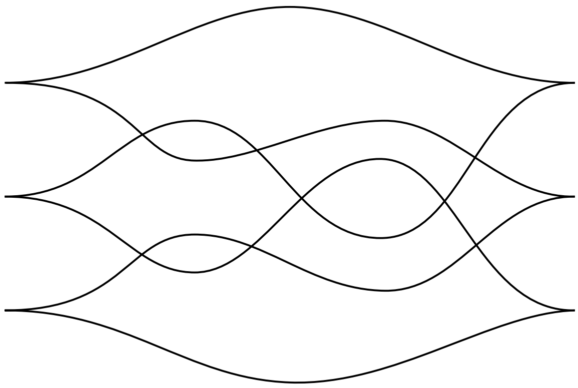

Figure 10: Front projection of $m(8_{21})$.

m(8_{21}) from a computational perspective

- **Braid encoding:**

```
[2, 4, 2, 4, 3, 3, 2, 4]
```

- **Number of augmentations of this front:** 16
- **Size of the matrix before compression:** 16×16
- **Number of equivalence classes of augmentations:** 10
- **BLCH matrix:**

$$\begin{pmatrix}
t^{-1}+4+2t & t^{-1}+3+t & t^{-1}+3+t & t^{-1}+3+t & 2+t & 2+t & 2+t & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+3+t & t^{-1}+4+2t & t^{-1}+3+t & t^{-1}+3+t & 2+t & 2+t & 2+t & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+3+t & t^{-1}+3+t & t^{-1}+4+2t & t^{-1}+3+t & 2+t & 2+t & 2+t & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+3+t & t^{-1}+3+t & t^{-1}+3+t & t^{-1}+4+2t & 2+t & 2+t & 2+t & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & 2+t & 1 & 1 & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & 1 & 2+t & 1 & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 & 1 & 1 & 2+t & t^{-1}+2 & t^{-1}+2 & t^{-1}+2 \\
2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 1 & 1 \\
2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 1 & 2+t & 1 \\
2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 2+t & 1 & 1 & 2+t
\end{pmatrix}$$

It was shown in [11, Section 3] that the Chekanov-Eliashberg DGA of K has 16 augmentations, which split into a set A of 4 augmentations and a set B of 12 augmentations such that

$$P_{K,\varepsilon}(t) = t^{-1} + 4 + 2t \text{ if } \varepsilon \in A$$

and

$$P_{K,\varepsilon}(t) = 2 + t \text{ if } \varepsilon \in B$$

This implies that augmentations in A are not DGA homotopic to augmentations in B. However, the number of DGA homotopy classes of augmentations for K was not determined in [11] as LCH does not suffice to obtain this information.

Using the BLCH criteria, the DGA homotopy classes can be determined systematically. It turns out that the augmentations in A are pairwise not DGA homotopic, because the Poincaré polynomial of any such pair of augmentations is $t^{-1} + 3 + t$. On the other hand, the set B splits into 6 DGA homotopy classes of augmentations.

Remark 22 : Actually, the braid encoding present in the data file was

```
[4, 3, 2, 1, 1, 5, 3, 2, 5, 4, 4, 4, 3, 2, 5, 4]
```

This encoding is not optimal, there are more crossings then necessary. This is due to automation in the standardization process of the Legendrian front, which causes fronts to be sometimes unnatural, with extra crossings.

There were 64 different augmentations and the matrix before compression was 64×64.

However, since these two braids correspond to Legendrian isotopic knots, the same BLCH matrices were obtained in both cases, with 10 equivalence classes of augmentations. The order of the rows and columns was different though: the BLCH matrix associated to the more "natural" braid encoding also gives a more natural ordering in the BLCH matrix (the previous discussion shows that "similar" augmentations are grouped together).

5 Further perspectives: the product structure on BLCH

If K is a Legendrian knot, let us denote by $C_n(K)$ the $\mathbb{Z}_2$ vector space freely generated by monomials of degree n and

$$C(K) = \bigoplus_{n\in\mathbb{Z}} C_n(K)$$

Given two augmentations ε_1 and ε_2, we have defined a differential

$$C_n(K) \xrightarrow{\partial^{\varepsilon_1,\varepsilon_2}} C_{n-1}(K)$$

BLCH was then obtained as the associated homological notion.

In terms of cohomology, we have a map

$$C^n(K) \xleftarrow[\mu^1_{\varepsilon_1,\varepsilon_2}]{} C^{n-1}(K)$$

There is more structure over $C(K)$. Given an integer $d \geq 1$ and augmentations $\varepsilon_0, \cdots, e_d$, one can define a map

$$\mu^d_{\varepsilon_0,\dots,\varepsilon_d} : C(K)^{\otimes d} \to C(K)$$

Definitions and properties can be found in [1], where μ was first defined. In fact, μ defines an $\mathcal{A}_\infty$-category whose objects are augmentations.

μ^2 provides a product structure:

$$BLCH_{\varepsilon_0,\varepsilon_1}(\Lambda) \bigotimes BLCH_{\varepsilon_1,\varepsilon_2}(\Lambda) \xrightarrow[\mu^2_{\varepsilon_0,\varepsilon_1,\varepsilon_2}]{} BLCH_{\varepsilon_0,\varepsilon_2}(\Lambda)$$

In particular, by taking $\varepsilon_0 = \varepsilon_2$, we have a map

$$BLCH^n_{\varepsilon_0,\varepsilon_1}(\Lambda) \bigotimes BLCH^{-n}_{\varepsilon_1,\varepsilon_0}(\Lambda) \xrightarrow[\mu^2_{\varepsilon_0,\varepsilon_1,\varepsilon_0}]{} BLCH^1_{\varepsilon_0,\varepsilon_0}(\Lambda)$$

There is a duality between $BLCH_{\varepsilon_0,\varepsilon_1}(\Lambda)$ and $BLCH^{-n}_{\varepsilon_1,\varepsilon_0}(\Lambda)$. Since $BLCH^1_{\varepsilon_0,\varepsilon_0}$ contains the fundamental class $[S^1]$, a natural question is to ask whether μ^2 identifies exactly the duality relation:

Conjecture 23. *Let $a \in LCH^n_\varepsilon(\Lambda)$ and $a^* \in LCH^{-n}_\varepsilon(\Lambda)$ be its dual, then*

$$\mu^2(a, a^*) = [S^1]$$

Remark 24 : Does not seem to hold in the case of not equivalent augmentations.

However, μ^2 is not known for almost any "interesting" example. To study further the properties of μ^2 (e.g. to investigate the previous conjecture), it is important to first investigate how it behaves on examples.

This can be done computationally: in terms of Morse complex sequences, determining the product structure boils down to counting chord paths with one corner: the necessary combinatorial work is already done in [9].

The related programming task and the analysis of computed results still needs to be done. So this is not the end of the BLCH story!

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